International Journal of Contemporary Educational Research (IJCER)

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To cite this article:

Uygun, T. & Guner, P. (2019). Representation of algebraic reasoning in sets through argumentation. *International Journal of Contemporary Educational Research, 6*(2), 215-229. DOI: https://doi.org/10.33200/ijcer.557781

IJCER International Journal of Contemporary Educational Research
Volume 6, Number 2, December 2019, 215-229 ISSN: 2148-3868

Representation of Algebraic Reasoning in Sets through Argumentation

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Abstract

The purpose of the current study is to examine the ways in which preservice middle school mathematics teachers (PMSMT) apply and represent algebraic reasoning in their solution processes for the problems in the concept of sets. This model provides detailed information about the reasoning made through the process of the solution of set problems. The study group of this case study was composed of 20 preservice mathematics teachers. The data were collected through written documents and whole class discussions. Based on the findings of the study, three ways to represent algebraic reasoning in sets emerged; context-based representation of algebraic reasoning, generalization-based representation of algebraic reasoning and formulization-based representation of algebraic reasoning. These ways determined based on the argumentations that they formed. They produced different warrants since they reasoned differently.

Key words: Algebraic reasoning, Argumentation, Representation of algebraic reasoning, Sets.

Introduction

When the mathematical concepts taught to different grade level of students are considered, they necessitate to improve algebraic reasoning. Improving students' algebraic reasoning is one of the most important responsibilities of mathematics teachers (Kaput, 1999; Schifter, 1999). These teachers are expected to have sufficient experience with rich and relational aspects of algebraic reasoning to perform their teaching in their classrooms. Hence, it is important that the teachers equipped with necessary knowledge and skills are to be educated in their preservice years to design their mathematics lessons particularly teaching concepts about algebra effectively. Through preservice years, teacher candidates are provided opportunities to acquire necessary knowledge and skills about content in order to perform teaching effectively when they become teachers in the future (Chapman, 2007; NCTM, 2006). In this respect, it is important to analyze preservice middle school mathematics teachers' (PMSMT) algebraic reasoning and the ways and representations on the concepts in detail and different perspectives. In the current study, it was paid attention on the knowledge, ways and representations used for making inferences about the PMSMT's algebraic reasoning. More specifically, the present study focused on how the PMSMT represent their algebraic reasoning on a particular mathematical concept of sets. The process was delimited on the sets because of various aspects of algebraic reasoning. In other words, algebraic reasoning is composed of various types of thought and comprehension of symbolism such as generalization by arithmetic and patterns, symbolism, understanding the structure of number system and patterns, and mathematical modelling (Kaput, 1999).

The preservice middle school mathematics teachers (PMSMT) are expected to design mathematics lessons providing opportunities for students to attain necessary knowledge and skills (Chapman, 2007; NCTM, 2006) particularly algebraic reasoning when they become inservice teachers (Blanton & Kaput, 2005). With this motivation, they need to have necessary knowledge about algebra concepts and skills as algebraic reasoning. This case should be explored in the particular concepts in detail so the mathematical concept of set has been chosen for the current study. When the nature of sets and types of thought about algebraic reasoning, and comprehension of symbolism as explained above by Kaput (1999) are considered, it can be stated the mathematical concept of sets providing a beneficial way to support the unity in mathematics language (Donmez, 2002) are appropriate to extract algebraic reasoning. The PMSMT need to understand and use generalization, symbolism and structure of number systems related to algebraic reasoning through understanding sets making

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operations on sets. When the sets in mathematics is thought, it can be claimed that the algebraic reasoning takes an important place. Even, the sets are taught enhancing algebraic reasoning at the necessary level. The primary school students are provided opportunities to engage in the sets with defined elements. Then, they are expected to grasp the connection between different sets to form the union or intersection of them. After that, they make a generalization about the intersection and union of sets. When necessities for the concept of the sets in higher grade levels are examined, the importance of algebraic reasoning increases. It can be explained that the algebra of sets is algebra of numbers illustrated with the help of the set-theoretic analogue. For example, sets suggests a way for the algebra of the set-theoretic operations of union, intersection and complementation, equality and inclusion (Courant, Robbins, & Stewart, 1996). In this respect, it is important to examine the algebraic reasoning in the mathematical concept of sets. This examination can be effectively made through the course that they learn sets and attain the ability of reasoning algebraically in sets. With this motivation, the present study was organized based on the purpose of identifying the PMSMT's representations of algebraic reasoning about the concept of sets.

This process could be examined through their ideas and argumentations effectively since the algebraic reasoning has been important to understand the concept. Studies in this field (Uygun & Akyuz, 2019; Conner et al., 2014; Krummheuer, 1995; Toulmin 1958/2003; Yackel, 2002) have showed that argumentation can be effectively used to analyze and comprehend mathematical aspects of social events in classrooms. Particularly, argumentation has facilitated the identification of kinds of mathematical reasoning focusing on their elements and distinctions among them in classroom conversations (Conner et al., 2014). Based on these perspectives, the ways of representation of algebraic reasoning and differences between them may be identified. Moreover, the reasons behind these differences may be determined so that mathematics educators may help preservice teachers to develop their reasoning and understanding about the concepts. Also, preservice teachers may make decisions about their profession related to algebraic reasoning being aware of different ways of reasoning about a particular concept. In this article, it was aimed to demonstrate that how argumentation could provide insight and enhance the representation of algebraic reasoning in different ways about the particular mathematical concept of sets. In this respect, the purpose of the present study was to examine the ways of how PMSMT represent their algebraic reasoning in a particular concept of sets through their argumentations. Hence, the answer of the question of "How was algebraic reasoning represented through solving problems and mathematical ideas used in the context of sets by the PMSMT?" was examined in the current study.

Theoretical Framework

Algebraic reasoning

The algebra has connection with many concepts such as sets, number, place value, basic facts and computation, operation concepts, proportional reasoning, measurement, geometry and data analysis (Kaput, 1998/1999). Moreover, it is important in expressions with symbols and making the extensions in numbers beyond the whole numbers in order to analyze the equations and functional relations, and determining the structure of the representational system including mathematical expressions and their connections. Furthermore, algebraic reasoning includes more actions than knowing the facts and techniques on sets. It is a way of thinking. Kaput (1999) emphasizes this view by explaining five kinds of thoughts and use symbolism including comprehension of symbolism such as generalization by arithmetic and patterns, symbolism, understanding the structure of number system and patterns, and mathematical modelling. This thinking as algebraic reasoning includes two central themes; making generalizations and using symbols to represent mathematical ideas and to represent and solve problems (Lew, 2004). Algebraic reasoning is defined as the ability to pay attention the quantities changing in the contexts and describe the ways in which they are connected. It also includes describing the change and the rate of this change benefiting from the tables, graphs, symbols, mathematical expressions, by thinking across and focusing on the relationship among those representations for particular contexts. Moreover, it necessitates to analyze and understand the algebraic expressions represented in multiple ways by considering their connections with the contexts that they formed and expressed (Carraher & Schliemann, 2007). Because of the role of these actions of algebraic reasoning in learning and understanding mathematics, it is important to develop algebraic reasoning. Students attain algebraic reasoning by connecting the concepts, realizing the relations and making generalizations. Moreover, it can be explained that if students reason algebraically, they can understand mathematical concepts such as patterns, and functions; represent and analyze mathematical situations using mathematical symbols; benefit from modelling to illustrate the change on the variables defined in the contexts (Kaput & Blanton, 2005; Van de Walle, Karp, & Bay-Williams, 2011).

Argumentation in Mathematics Education

The present study is organized based on the relationship between argumentation and reasoning. In many contexts, argumentation is described as "trains of reasoning" (Toulmin, Rieke & Janik, 1984, p.12). Also, Conner, Singletary, Smith, Wegner and Franncisco (2014) explain that creating an argument and reasoning are performed together enhancing the occurrence of each other. Moreover, they are examined and expressed through similar processes in mathematics (Conner et al., 2014). Argumentation can be expressed as a kind of discourse formed through justification, association and use of ideas (Ibraim & Justi, 2016). It can also be explained as the conviction mechanism used in a conflicted environment; a social and intellectual process involving verbal activities aimed at supporting or refuting an idea; a learning environment activity in which ideas that support conceptual meaning are structured (Binkley, 1995; Duschl & Osborne, 2002; Ohlsson, 1995; Siegel, 1995; Van Eemeren, 1995). It is the process of supporting and defending an idea. When it comes to mathematics, argumentation focuses on how mathematical ideas of individuals and why they are supported are related to one another and how they are used in discussions or in communication (Pedemonte, 2007).

Through algebraic reasoning, it is important to justify mathematical expressions since students focus on the reasons about concepts. In this respect, argumentations can be useful to be linked with algebraic reasoning. Argumentation encourages the development and formation of mathematical justifications in a social learning environment in which the students share and criticize others' ideas and explanations (Uygun & Akyuz, 2019; Yackel & Cobb, 1996). In this respect, the students reason the ideas so that they can understand and learn the concepts through argumentations (Lampert, 1990). Through argumentations, students do not apply the rules and theorems to the contexts by memorizing (Uygun & Akyuz, 2019; Pedemonte, 2007). Instead of this case, they explain how and why they apply them in the problem situations. Therefore, the argumentations are useful to develop reasoning and also algebraic reasoning. Moreover, argumentations enhance conceptual change, understanding of the concepts and solution of the problems for the students in social learning environment (Abi-El-Mona & Abd-El-Khalick, 2011; Jonassen & Kim, 2010). Moreover, classroom discussions can be useful to encourage students' development of algebraic reasoning (Kızıltoprak, & Yavuzsoy-Köse, 2017). In that respect, argumentation as a kind of mathematical discourse formed through classroom discussions can help preservice teachers improve their algebraic reasoning.

Method

The present study was organized based on case study research design due to its help of understanding processes involved in the study thoroughly (Merriam, 1998). Case study is useful to examine a phenomenon in detail in its real life context and holistic perspective (Creswell, 2012). Different students' responses on a particular issue such as sets were examined as case for the current study. In this respect, preservice middle school mathematics teachers' representation of algebraic reasoning was examined and reported based on its meaningful characteristics.

Participants

Purposive sampling strategy was used in order to select the participants. The judgment about determining the participants for the present study was being enrolled in the undergraduate course of "General Mathematics" since they are taught sets in this course. Then, 20 preservice mathematics teachers who were freshmen and enrolled in this course, were selected as participants for the present study. Also, they were the candidates of secondary school teachers. Of these PMSMT participating in the study, 12 of them were male and 8 were female. All of them were asked some open-ended problems related to sets and operations on sets such as union and intersection and needing algebraic thinking. Then, they discussed about what they did to solve them and how they reasoned.

Data Collection

Three-week instructional sequence and two hours in each week were conducted to the PMSMT. This sequence were designed about the concept of sets. In this sequence, the PMSMT engaged in activities and problems formed by literature review and textbooks about sets. The textbooks about sets and the problems about them were examined and mostly used operations and examples were determined. Then, considering these determined problems, three tests including almost five open-ended questions about sets and operations on them to be used in each week were prepared by the researchers. Moreover, these problems (see Appendix) were determined by considering that they should encourage reasoning about them, making explanations by generalizing, understanding symbolism considering five types of thought explained by Kaput (1999). The problems on the tests were examined by an academician having doctorate degree in mathematics and two PMSMT not participating in the study. By their views about the problems, the tests were revised and re-designed to be used through three-week instructional sequence. Then, each of the tests were conducted to the participants of the study respectively based on order of concept and operations of sets. The data were collected through written documents and whole class discussions. Initially, they were asked to solve problems on tests by justifying the correctness of set expressions explained in the problems. While they were solving these problems and after the solution process had been completed, they participated in whole class discussion. In this discussion process, the instructor observed the studies of the PMSMT and then, initiated the discussions focusing on different ways, missing and incorrect parts of the solutions. They talked about what they did and why they did so on their written documents. Also, the data collected through written documents were firstly examined through observing the PMSMT while they were studying about the problems on the tests. The ideas determined and not understood were asked and criticized through discussions. Whole class discussion process was recorded by two video cameras; one camera for in front of the classroom and one for back of the classroom. In order to collect detailed data about the events and discussion happened in the instructional sequence and to make correct inferences, two video cameras were used. Through instruction, the instructor was not able to observe all of the participants effectively at the same time. Hence, by video recordings, the roles and actions of all of the participants were recorded by video cameras so that the analysis could be performed effectively. The participants were aware of and aim of usage video cameras in the classrooms. Also, in order to remove the effect of video camera on the PMSMT for the current study, the video cameras were placed and used in the classroom two weeks before data collection process. After the discussions were completed, they were transcribed verbatim.

Data analysis

In classroom conversations, the PMSMT'S ideas and argumentations represent their reasoning about their understanding and algebraic reasoning. The signs used in this process to explain their ideas are considered in a broad sense, including written or linguistic terms, gestures (Arzarello 2006; Ernest 2008; Radford 2002). Moreover, they are considered as constitutive parts of reasoning. With the help of the ways followed by the students and the interpretations of them for the students, the ways of representation and use of algebraic reasoning can be understood beneficially. The video recordings were transcribed and coded. The instructor's explanations were illustrated by the letter of R and the PMSMT's explanations were done using S where n showed different positive natural numbers representing different PMSMTs. The data analysis process also included two parts; document and whole class discussion analysis. In the process of qualitative data analysis, Toulmin's model of argumentation was used. The core of this model is composed of three elements; the data, claim and warrant. In the model, the learners may provide encouraging explanations for the warrant and argument in the form of backing, rebuttals and qualifiers. The element of claim is related to conclusion statement of the argumentation. The data refers to the evidence and warrant as the way linking the data with the proposed conclusion. Also, backing supports the warrant itself (Stephan et al., 2003). Initially, the components of Toulmin's (1958/2003) model were identified in the transcribed whole class discussions. Then, the core elements of model were used in order to determine the codes and themes. The codes for the core elements of the model are content, goal of formation, condition and characteristic. The themes formed by the Toulmin's model and the codes were organized as context-based representation of algebraic reasoning, generalization-based representation of algebraic reasoning and formulization-based representation of algebraic reasoning. The transcripts were analyzed in order to determine argumentation logs representing the mathematical ideas including argumentation elements such as claim, data, and warrant. Each argumentation log for a mathematical idea was identified as code. Related codes were explained by themes as in Table 1. Moreover, the trustworthiness was supported by the investigator triangulation performed by two researchers of the study. The researchers analyzed the data independently and then compared their analysis. They made comparisons among the codes and themes that they determined independently. Then, they formed a list of code and themes. The similar codes and themes were placed to the list. They discussed about the different codes and themes until they reached consensus. Through discussion, they formed common codes and themes for different ones. Their agreement about the codes and elements of Toulmin' model of argumentation by the data was determined approximately 85%. At the end of the analysis, a different researcher who was academician with sufficient knowledge about algebra read and assessed the analysis part of the study considering the properties of consistency and coherence. By doing investigator triangulation and peer debriefing, the trustworthiness of the present study was satisfied successfully (Lincoln & Guba, 1985).

Table 1. *Codes and themes*

Codes	Themes
Not considering generalization Using limited symbolism or symbols for representation Focusing on particular context rather than structure of operations	Context-based representation of algebraic reasoning
Considering limited generalization Using explanations rather than symbolism Focusing on meaning of structure of operations	Generalization-based representation of algebraic reasoning
Considering generalization Using symbolism effectively Focusing on structure of operations by mathematical language	Formulization-based representation of algebraic reasoning

Findings

Representation Ways of Algebraic Reasoning in Sets

When the process of the PMSMT's engaging in set expressions were examined based on argumentation framework, the ways of representation of algebraic reasoning in sets made by PMSMT were separated into three groups. Hence, three types of representation of algebraic reasoning were evolved through the argumentations.

Context-based Representation of Algebraic Reasoning

In this way of reasoning, the PMSMT focused on representation of the expression so they formed Venn diagrams and shaded regions given in the set expressions. Usually, the PMSMT started by drawing representations and forming specific contexts. As illustrated in Figure 1 and Figure 2, they drew Venn-diagrams by coloring the parts explained right and left parts in the equation. Figure 1 represented left-hand side of the set expression and Figure 2 did the right-hand side of the expression. The PMSMT using this way of representation performed similar actions in other questions.

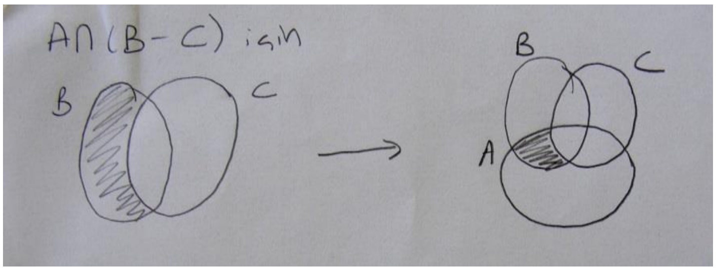


Figure 1. The representation of $A\cap(B - C)$ of the S_1

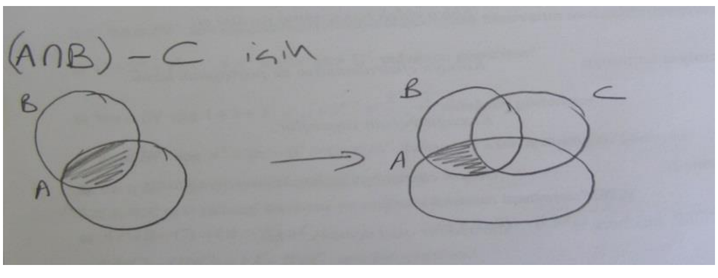


Figure 2. The representation of $(A\cap B)–C$ of the S_1

The PMSMT represented the set expressions explained in the equation. In that solution process, initially, the PMSMT made reasoning by shading on diagrams such as Venn-diagrams. Also, they explained that they used the shapes to understand and to describe the problem clearly and to compare the formed shapes. Some part of the argumentation happened as follows:

R: Why did you draw the Venn-diagram?
S_1: Since, I wanted to see which regions belonged to $A \cap (B–C)$ and $(A \cap B)–C$ in Venn-diagram. I determined their regions based on the operations. Intersection set is formed by common elements so…[*DATA*]
R: Why did you want to see their regions?
S_1: By seeing their regions, I realized that they represented the same parts in the diagram. So, I was sure that the statement was true. [*CLAIM*]
R: What is the benefit of becoming sure about the equation by using the diagram?
S_1: I make this since when I become sure about the truth of explanation, I knew that I can continue to plan the solution.
R: What was your plan for this question?
S_1: While representing the different sides of the equation, I followed different steps but attained the same part in the diagram. For example, while representing left-hand side of the equation, I determined the place of $(B – C)$ first and then, the place of the intersection of the set of A and the set of I determined in the previous step. Moreover, while solving the other side of equation, in the first step, I determined the place of $(A \cap B)$ and then I found the place of the difference of set that I determined in the first step and the set of C. I understood that I could solve the question by following similar steps. [*WARRANT*]

As understood from this episode of the argumentation, S_1 claimed the truth of the expression and she needed to see the representation of the set expression to understand the problem. For this claim, she used the Venn-diagrams and the definitions of the operations in its written form for the data of the argument. In other words, she used the expressions of the definitions of the operations but not mathematically. For example, intersection of two sets was composed of the common elements of these sets. Then, she made comparison between right and left-hand sides of the equation. By making these representations, she obtained information about the steps to follow in order for solving the questions as the warrant of the argument. In this solution process, the students solved the question by explaining and representing that both hand-sides of the equation were same. In order to help PMSMT represent set expressions by using standard symbolic form, they were asked to make further mathematical explanation. Then, they focused on the contexts and formed the particular sets to justify the truth of set expressions.

As illustrated in Figure 3, some PMSMT formed specific sets with particular elements and then they solved the problem by using roter notation. In that solution process, they used symbols (-, A, B, C, $\cap$, 1, 2, 3, etc.) and some of these symbols such as numbers selected randomly by them. They tried to understand and describe the expression by using roter notation and specifying the explanation into a specific context. At the end, they made comparison of the attained sets for both hand-sides of the set expression. Also, argumentation was produced by focusing on the solution of one of them for these activities as follows:

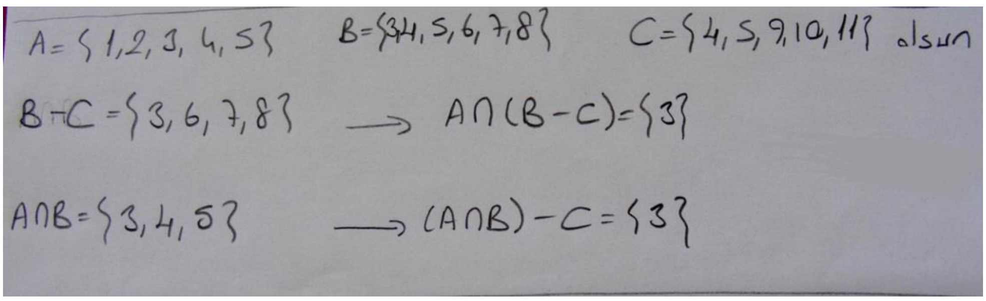


Figure 3. Representation of set formed by S_4

R: What did you do in this part?
S_4: There are three sets as A, B and C in the equation. I assigned the elements to these sets.
R: Why did you do so?

S_4: Since, I wanted to see a clear example to understand it. At first glance, I could not clearly understand what the equation means.
R: What did you do after assigning the elements to the sets?
S_4: After forming the sets with elements, I made the operations represented as $A\cap(B–C)$ and $(A\cap B)–C$ [WARRANT]. Then, I attained the same sets. By following different steps with the same sets, I attained the same sets. I understood that $A\cap(B–C)$ and $(A\cap B)–C$ represented the same sets. The equation is correct [CLAIM].

As understood from the dialogue that the Student 4 assigned randomly selected elements for the sets. He formed particular A, B and C sets. He used these sets and the definitions of operations on set expressions in the equation as data for his argument. Then, they represented the expression by using elements of these sets. He understood the problem and solved it in this way. He represented his reasoning based on a specific context. Also, he made comparison between right and left-hand sides of the equation. Hence, by making comparison through following the operations on these sets, he proposed the warrant of his argument. By forming these sets, they obtained information about different written forms of the specific set. He became aware of that he needed to prove that the different sides of the equation are different written forms of the set. Hence, he produced the claim that the equation or mathematical expression is correct.

When the arguments were examined, the PMSMT produced their claims benefiting from the data and warrant supported by shading regions or particular sets and the expressions of the operations. As it was understood from these episodes of the argumentations, the PMSMT tried to understand the questions and to make reasoning by forming specific examples. They formed contexts for set expressions and made set operations based on these contexts. Hence, this way was named as context-based representation of algebraic reasoning. Moreover, when the documents and arguments were clearly examined, it could be said that the PMSMT dealt with the procedures and tried to increase their procedural knowledge by using context-based question formed by them. In both cases, the PMSMT were asked to justify the equality of both hand-sides of the equation but they tried to show this equality based on operations without proving. Moreover, instead of considering the sets as arbitrary sets including unknown elements, they formed the sets with limited elements based on their thoughts or the regions covered on the diagrams by focusing on operational conception in the expression by producing contexts.

Despite its apparently concrete nature based on representations and contexts, context-based representation of algebraic reasoning does not mean reflecting mathematical thinking in a simple way. As it was shown on the figures above and formed arguments, this type of representation for algebraic reasoning comes up to perception becoming evolved mechanism at high level and rhythmic coordination of words, representations, formation of contexts and symbols. In this process, it was observed that the PMSMT had the understanding of structural features of sets and set expressions. Also, they focused on graphical and symbolic representation forms about sets and set operations but they could not successfully use them. Because, they could not accurately consider about generalization and standard representation form of the set expression.

Generalization-based Representation of Algebraic Reasoning

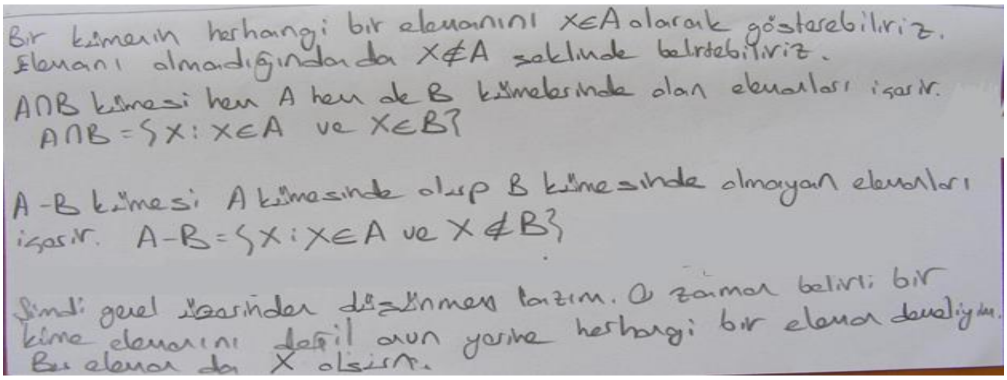
In this way of representation of algebraic reasoning, the PMSMT benefited from the systems of mathematics and language. The students used the symbols only in the mathematical definitions of the operations of sets. The PMSMT used their understanding of set theoretic operations and then they made trials to transfer this context-based representation of algebraic understanding to standard algebraic understanding. They used the formal definitions of operations on sets in this way of representation of algebraic reasoning. However, they could not successfully transfer and use these definitions in the operations asked in the problem.

In this way, they benefited from general definitions of operations of sets in the transition process as illustrated in Figure 4. In the problem, there existed intersection and difference operations of sets. S_7 produced formal definitions of these operations. Using her expressions and mathematical representations, she considered about generalization aspect of the set expressions in the problems by the structures of set operation. Hence, she used this understanding and produced her solutions to form standard expression. However, she could not appropriately use related symbols and notations in her solutions. These activities were also asked to the students. There was a typical argumentation for the claim about the truth of equation related to set expression as follows:

R: What did you do in this part of your solution?
S_7: I used the definitions of set union and set difference written in the problem [*DATA*] in order to show the truth of the equation [*CLAIM*].

R: Why did you use the letter of X here?

S_7: I need to prove the statement for arbitrary sets. I use the letter of x for any element in the set. I mean by using this letter that x is an arbitrary element in the set so that the operations become valid for all elements in the set. Also, as I remembered from the lesson, if an object belongs to a set, it is said to be an element of the set. For example, if x is an element of set A, we write $x \in A$, and if x is not an element of A we write $x \notin A$. The notations are important in doing mathematics since by using right notations, we can do mathematics. [*WARRANT*]



We can show an arbitrary element of a set as $x \in A$ for. We can show non-elements of the set as $x \notin A$. $A \cap B$ set shows the common elements of the sets of A and B.
$A \cap B = \{x: x \in A \text{ ve } x \in B\}$
The set of A-B shows the elements in the set of A and they are not also the elements of the set of B. $A\text{-}B = \{x: x \in A \text{ ve } x \notin B\}$
Now, it is important to think in general perspective. Hence, I need to write an arbitrary element instead of a particular element.

Figure 4. Written form of warrant explained by S_7

R: Why did you use the definitions of operations?

S_7: Actually, I did not comprehend how to prove the equation formally. Hence, I wrote the formal definitions of related operations. Then, I tried to prove the statement by using notations in formal way.

As understood from the documents and the discussions, in this way of representation of solution process, the PMSMT had structural knowledge about sets and they were aware of the case of generalization in set expressions. However, they could not use standard formal way of algebraic reasoning about sets. Hence, they focused on verbal explanations about solution of set expression for this argument. In this process, the PMSMT using this way engaged in the mathematical definitions of the operations in the equation and made some formal manipulations on the formulas and procedures. They examined them and then, they tried to represent the expression in standard algebraic form.

Formulization-based Representation of Algebraic Reasoning

Justifying the expression by using the algebraic standard symbolism might be much more difficult than expressing it in words whatever their grades are. This way of representing algebraic reasoning in sets could be achieved by the PMSMT by putting much effort on it. In this way of the representation of algebraic reasoning in solution process, it was observed that the PMSMT had structural knowledge about sets and set operations. Also, they could appropriately use standard symbolic forms about set expression. They used formal right algebraic symbols (x, A, B, C, -, $\cap$) by understanding their meanings and roles as illustrated in Figure 5.

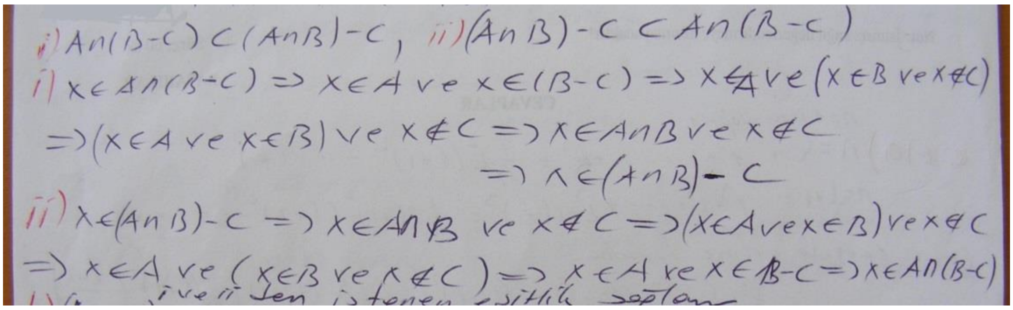


Figure 5. Solution of S_{16}

As seen from Figure 5, they could effectively justify set expression by using standard algebraic notation by explaining the claim about the truth of the mathematical expression in the problem. In order to clearly examine the way of representing their algebraic reasoning, their argumentations were examined. Some episode of this argumentation is as follows:

R: What did you do in this part of your solution?

S_{16}: Initially, I need to show that I proved that two sets on different sides of equation are equal sets [*CLAIM*] by demonstrating that they are subsets of each other. In this case, I showed both $A\cap(B–C) \subseteq (A\cap B)–C$ and $(A\cap B)–C \subseteq A\cap(B–C)$. Then, I chose x as an arbitrary element in the set and wrote $x\in A\cap(B–C)$ [*DATA*]. Afterwards, I examined the possibilities about the existence of x in the sets of A, B and C. When I considered the property of the set intersection and set difference, x is the element of the sets of A and B but it is not the element of set C. Then, I continued to prove the equation. At the end, I found that both sides of the equation were same so that I proved it. [*WARRANT*]

In this argument, S_{16} provided data by the definitions and properties of intersection, difference and subsets. Then, operational process and transfer of these properties were explained as the warrant of the argument. In the warrant, they thought in general perspective about set expression. They grasped the important point of algebraic reasoning by using x as an arbitrarily chosen element in a set and used it through the process. Without having any oppositions in the context, they began to use a new version of algebraic reasoning and progressed through the emergence of algebraic reasoning representing a deep region for algebraic reasoning. They progressed from a referential comprehension of signs (signs used to illustrate significant elements in sets). The participants could effectively justify the set expression by using the algebraic reasoning with formal notations. In this way, the PMSMT focused on representation of justifying the expression by using standard signs with mathematical expressions accurately. Hence, this way of representation of algebraic reasoning can be named as formulization-based representation of algebraic reasoning.

Transition between the Ways of Representation of Algebraic Reasoning in Sets

In the first way of representation of algebraic reasoning, the PMSMT tried to think about set expressions by using specific examples and representations or shading regions on diagrams. This way was named as context-based representation of algebraic reasoning since they made trials to justify the set expression by using contexts. In the second way, they tried to make actions to justify set expression. This justification process was not about a specific context or real life examples. They focused on generalization using structural knowledge about sets and set operations. However, they made verbal statements instead of using standard related symbols. So, this way was named as generalization-based algebraic reasoning. They thought about the general definitions and properties of the sets and operations and made actions to transfer structural knowledge to standard symbolic form. At the last way, they made actions to justify expressions by using standard algebraic and formal symbols. Hence, this last one was named as formulization-based algebraic reasoning.

Through the three-week instructional sequence, PMSMT engaged in the problems individually and then participated in whole class discussions. Through these discussions they examined their solutions and reasoning under the guidance of the instructor. Through argumentations, the PMSMT shared, analyzed and understood others' ways of representation of algebraic reasoning in sets. In this process, they realized the missing parts of their representation ways and usage of algebraic reasoning, and then made effort to form correct justification by understanding and using others' ways of representation of algebraic reasoning. Also, they improved their way of representation of algebraic reasoning in sets and made transitions between three ways emerged in the study. When the PMSMT's justification processes made in different set expressions in which they engaged in three weeks were examined, it was observed 5 of PMSMT represented two ways of representation of algebraic reasoning and 2 of them showed all ways in their processes. Moreover, 13 of PMSMT illustrated one of these ways of representation of algebraic reasoning as illustrated in Table 2.

Table 2. Number of PMSMT using different ways of representation of algebraic reasoning

Ways of Representation of Algebraic Reasoning	1	2	3	1→ 2	2 → 3	1 → 2→ 3
Number of PMSMT	2	4	7	2	3	2

1: Context-based representation of algebraic reasoning
2: Generalization-based representation of algebraic thinking
3: Formulization-based representation of algebraic thinking
1→2: Moving from context-based to generalization-based representation of algebraic reasoning
2→3: Moving from generalization-based to formulization-based representation of algebraic reasoning
1→2→3: Moving from context-based to generalization-based and then to the formulization-based representation of algebraic reasoning.

It can be claimed that 12 of the PMSMT illustrated formulization-based representation of algebraic reasoning by using formal signs to justify set expressions. While 5 of them achieved this by processing through different ways of representation of algebraic reasoning in order, 7 of them used formulization-based representation way of algebraic reasoning effectively by not using other ways. Moreover, 11 of PMSMT represented generalization-based algebraic reasoning. While 5 of them used this way as a stage to move toward the standard algebraic reasoning type, 2 of them reached this way coming from the first type and 4 of them directly represented this type in their solutions. Also, 6 of them illustrated syntactic algebraic reasoning type in their solutions. While 4 of them used this type as a step to move toward other types, 2 of them solved the questions directly using this way of representation of algebraic reasoning. Moreover, whole class discussion process about a problem occurred under the guidance of the instructor. For example, the instructor initiated the discussion by asking the problem about examining the truth of $A\cap(B–C) = (A\cap B)–C$ and wanted a PMSMT using the context-based representation of algebraic reasoning to explain the solution as follows:

> S_4: I assigned the particular elements to the sets and made the intersection and difference operations. Hence, I attained the sets having same elements for both hand sides of the equation.

For this part of the argumentation, S_4 explained the truth of the equation as claim correctly. Then, he provided data by the definitions of operations. He provided insufficient warrant but the mathematical expressions of the operations and their transfer to the solution had missing part. This missing part was removed by a PMSMT using generalization-based representation of algebraic reasoning. She proposed mathematical expressions of the operations and explained the necessity of choosing and arbitrary element rather than assigning particular elements to the sets. This PMSMT completed the missing part of the data but she could not provide the complete and accurate warrant of the argumentation. Afterwards, a PMSMT using formulization-based representation of algebraic reasoning proposed the solution of the problem as the warrant appropriately and sufficiently as in Figure 5.

Discussion and Conclusion

The results of the study showed that representation of algebraic reasoning is important to understand mathematical concepts such as sets. In this way, preservice mathematics teachers can effectively comprehend the sets, operations on them and their properties. Also, their understanding and knowledge about sets can be determined and improved. In the study, there are three ways of representation of algebraic reasoning emerged through justification processes of set expressions. Toulmin's (1958/2003) argumentation model enhances and provides the perspective for analyzing algebraic reasoning based on the PMSMT's understanding of sets. Based on the view that Toulmin's model provides a beneficial way to represent mathematical classroom interactions (Uygun & Akyuz, 2019; Brown, 2017; Conner et al., 2014; Prusak, Hershkowitz & Schwarz, 2012), it supports the identification of important parts of mathematical discussions in the classrooms in the algebraic perspective. Moreover, this model could provide to illustrate and examine the learning processes and comprehend the ideas, reasoning and explanations of the PMSMT. This finding can be supported by previous research in the literature. In other words, the previous research show that Toulmin's model of argumentation proposes a beneficial way to share ideas, to make explanation to convince other about the truth of the idea, to listen counter ideas and their justifications and provide evidences for the explanations (Knipping, 2008; Krummheuer, 1995). Hence, the processes of formation of algebraic reasoning and the differences between them could be examined in detail. After analyzing data by the Toulmin's model of argumentation, it was observed that the PMSMT used similar claim and data for the same problems. However, they linked data and claims by using different warrants because they reasoned and solved differently. In this respect, it could be stated that warrants would be important to focus on various reasoning ways of the PMSMT, how they represented and used their reasoning, and how they differentiated. As the bridge between the data and claim, the warrants necessitates to reason to make this connection and form meaningful argument. This finding can be encouraged by the results of the previous research in the literature examining the roles of warrant in mathematics education (Alcock & Weber, 2005; Conner, 2012; Rodd, 2000).

Generally warrant represents the solution processes of a problem. Based on this view, the warrants represented the PMSMT's usage of algebraic reasoning to justify set expressions and representation of algebraic reasoning. Hence, by focusing on the warrants of the argumentations, differences between the representation and usage of algebraic reasoning about the problems of sets were examined. The warrants provided by them were classified under three different titles as the themes of the analysis process. Hence, three ways of representation of algebraic reasoning on the sets emerged; context-based representation of algebraic reasoning, generalization-based representation of algebraic reasoning and formulization-based representation of algebraic reasoning. Afterwards, the PMSMT's representation of algebraic reasoning in the concept of sets was examined by these themes.

In the first way, context-based representation of algebraic reasoning was used to justify expressions. They formed specific contexts about expressions and used representations such as diagrams and formed particular contexts. The engagements of them using contexts can be linked with operational conception of algebraic reasoning included in the model of Sfard and Linchevski (1994) formed for the development of algebraic reasoning because the main focus point is on the operations. Moreover, this type might also be linked to the structuring layer of Harel (2008) because the learners focus on arithmetical structures and symbols. However, the usage of Venn-diagrams and shading regions have been examined in previous research and they have stated the important place of visualization in problem solving and explaining reasoning process (Hodgson, 1996; Giordano, 1990; Tall & Thompson, 1989).

In the second way, PMSMT used generalization-based representation of algebraic reasoning. In this way, they used and examined the formal definitions of main and simple set operations related to set expressions. By using these definitions, they tried to justify set expression using verbal expressions but could not effectively use standard related notations. At that point, their usage of verbal explanations in their expressions can be related to the effect of use and interpretation of natural language to understand and learn the concepts. This view can be encouraged by the researchers (Pimm, 1987; Walkerdine, 1988). They made some manipulations on signs, formulas and explanations. This manipulation is important to represent standard algebraic reasoning. Hence, it should be encouraged carefully since these manipulations performed on the signs necessitates to make changes on intention so that it becomes possible to focus on the signs meaningfully. Moreover, it can be added that the PMSMT could perform abstract manipulation of signs by attaching new meanings considering the rules as the rules of a game as explained in previous research (Husserl, 1970). On the other hand, if these kinds of manipulations are not made carefully, the efforts put forward to obtain algebraic reasoning may be unnecessary as Russell (1976) explained the manipulations performed on the signs formally as explaining the reality in an empty way. This process can be linked with transition from operational to structural conception with respect to the model of Sfard and Linchevski (1994). Moreover, this way can also be linked with the generalizing layer of Harel (2008) since they focus on variables and common form of a set of formulas. Also, it can be explained that PMSMT using the second way of representation of algebraic reasoning have the properties of structuring layer previously and have been attaining the properties of generalizing layer.

The last way of representation of algebraic reasoning emerged in the study is formulization-based representation of algebraic reasoning. They appropriately justified expressions by using formal algebraic notations. They could effectively generalize their structural knowledge about sets using standard related signs. It can be concluded that PMSMT could justify expressions so this can be linked with the structural dimension of algebra told by Kieran (1990), Kirshner (2001), Hoch and Dreyfus (2006). It can be explained that the participants attained structural conception in formal way in the model of Sfard and Linchevski (1996). Formulization-based algebraic reasoning can be linked with structural conception since structural conception is defined as the approach condensing the information and broadening the views (Sfard & Linchevski, 1994). Moreover, this way can also be related to the representing layer of Harel (2008) since they focus on symbolic and formal form of mathematical expressions. It can be stated that PMSMT have attained the properties of structuring and generalizing layers previously and then they have begun to use symbolic and formal mathematical expressions as the representation system of algebra.

Based on the explanations above, it can be stated that Toulmin's (1958/2003) argumentation model enhanced the establishment of mathematical understanding. Through analyzing and identifying different representations of algebraic reasoning in sets, the PMSMT's mathematical knowledge was determined because the elements of the model focused on their usage of the knowledge. This explanation encourages the findings of previous research claiming the important role of argumentations on the establishment of mathematical knowledge (Rasmussen & Stephan, 2008; Uygun & Akyuz, 2019). The findings of the study show that the warrant has important role in identification of the differences between the representations of algebraic reasoning on sets. It does not only explain the difference, it also provides insight into the formation of algebraic reasoning. There exist distinctions between the warrants of the classroom interactions analyzed in the current study based on the explanations of Inglis, Meija-Ramos and Simpson (2007). They expressed the roles of warrant as reducing uncertainty and removing uncertainty. In the study, the warrant took the role as reducing the uncertainty in the context-based representation of algebraic reasoning since the PMSMT reasoned on the definitions of set operations and their usage on particular contexts were explained by the warrants. In the generalization-based representation of algebraic reasoning and formulization-based representation of algebraic reasoning, the warrant was used as removing the uncertainty since the PMSMT analyzed the reasons behind the set operations. Therefore, it could be stated that the Toulmin's model of argumentation gives insight into the identification and analysis of algebraic reasoning in classroom interactions. This finding encourages the explanation of Toulmin (1958/2003) and Inglis and colleagues (2007) about the warrants in mathematics education.

In the process of transition between representation ways of algebraic reasoning, most of the PMSMT follow all ways of representation of algebraic reasoning emerged in the study in order or from the second way to the third one or from the first way to the second one. When the topic of set is considered in our education system from secondary education to college level, it can be concluded that algebraic reasoning in sets is important since it is related to algebraic reasoning and the topics of the undergraduate course of Abstract Algebra (Hodgson, 1996). Therefore, it is important to help the PMSMT produce formal justifications for set expressions and they can be educated in a way explained in the present study. Moreover, while justifying set expressions and developing representation system for algebraic reasoning in sets, some PMSMT formed particular contexts and obtained specific knowledge about the context. By using main definitions, they generalized their contexts. It can result from the teaching style in our education system tendency of the PMSMT on moving specific examples to the general one. In our education system, specific examples are provided to the students by using specific numbers or quantities. Then, they are expected to make generalizations by grasping the common point between the examples (Baykul, 2012). This process maintains its existence in the process of sets. Even if the questions are organized and designed based on general terms, they can need to think and understand it in specific terms and contexts. Then, they can progress through the particular terms. In this respect, it may be said that whatever the grade levels are, the students may produce standard representation by forming contexts and then transferring it by using verbal statements in different ages in their life. Moreover, it was observed that the PMSMT could make transitions among different types of representations of algebraic reasoning. By developing their understanding, participating in discussions and solving problems, they could improve their representation of algebraic reasoning. This finding can be confirmed by the result of the previous research (Uygun & Akyuz, 2019; Jonassen & Kim, 2010; Staples & Newton, 2016). These previous research state that argumentations make the student improve their proficiency in practice and understandings about the mathematical concepts.

By the usage of Toulmin's model through analyzing algebraic reasoning on particular mathematical concepts as happened in the current study, a way or strategy has been provided for mathematics educators in order to explore the identification of algebraic reasoning performed by the PMSMT in the classroom conversations and focusing on the differences about how the PMSMT represent their algebraic reasoning. Hence, the mathematics educators can help the PMSMT develop their algebraic reasoning and represent their reasoning with accurate certainty in appropriate form. The usage of Toulmin's model on algebraic reasoning should be conducted to classroom interactions about different mathematical concepts and different grade levels so that perspective about the model in mathematics education can be provided effectively. For example, the middle school students' development of algebraic reasoning on the concept of patterns can be explored with the help of argumentation. Also, high school students' development of algebraic reasoning can be explored in the same way. In this respect, further research can be performed based on this suggestion. For example, this process can be repeated in computer-assisted learning environment. Different grade level of students' algebraic reasoning can be explored through their engagement in solving problem by related software and then they can discuss their solutions. Hence, their development of algebraic reasoning in a computer-based learning environment with the help of argumentation. Moreover, a further study examining the representation of algebraic reasoning on sets can be conducted to different grade level of students. Hence, the differences and similarities can be explored.

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Appendix

1. Show that $A \cap (B U C) = (A \cap B)\ U\ (A \cap C)$ [item of Test 1]
2. Show that $AU(B \cap C) = (AUB) \cap (BUC)$ [item of Test 1]
3. Show that $A \cap (B - C) = (A \cap B) - C$ [item of Test 2]
4. Show that $A-(BUC) = (A-B) \cap (A-C)$ [item of Test 2]
5. Show that $A-B = A \cap B^c$ [item of Test 3]
6. Show that $A-B = A \cap B^c = B^c - A^c$ [item of Test 3]